Minimal Model Program

Learning Seminar.

Week 5:

- Rationality Theorem
- Non-vanishing Theorem.

Rationality Theorem:

Lemma 1: Y a smooth projective variety, $D_1, \dots, D_n$ Cartier divisors on Y.
A normal crossing with $\lceil A \rceil \geq 0$.

$$P(u_1, \dots, u_n) := \chi\left(\sum_i u_i D_i + \lceil A \rceil\right).$$

Assume that for certain u_i, $\sum_i u_i D_i$ is nef and $\sum_i u_i D_i + A - K_Y$ ample.

Then, $P \neq 0$ of degree $\leq \dim Y$.

Proof: For $m \gg 0$, $\sum_i m u_i D_i + A - K_Y$ is still ample,

$$H^i\left(\sum_i m u_i D_i + \lceil A \rceil\right) = 0 \text{ for } i > 0 \text{ by KV vanishing.}$$

By Non-vanishing Theorem $h^0\left(\sum_i m u_i D_i + \lceil A \rceil\right) \neq 0$

so $\chi\left(\sum_i m u_i D_i + \lceil A \rceil\right) \neq 0$. Hence

$$P(m u_1, \dots, m u_n) \neq 0.$$

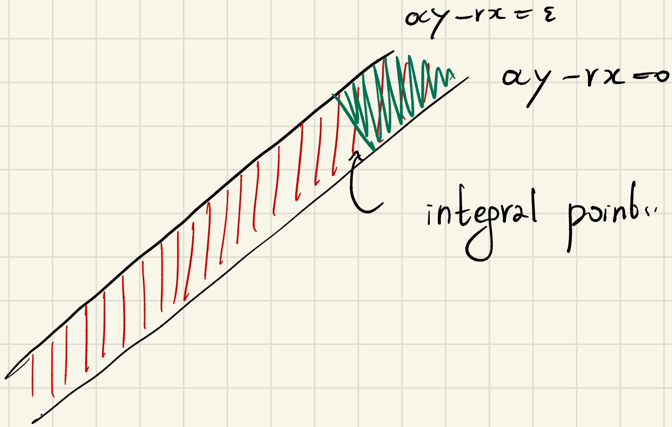
Lemma 2: Let $P(x,y) \neq 0$ polynomial of degree $\le \overset{\dim(X)}{\overset{\|}{n}}$

Assume P vanishes for all sufficiently large integral solution of

$0 < \alpha y - rx < \varepsilon$ for $\alpha \in \mathbb{Z}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$.

Then, r is rational and in reduced form it has denominator $\le \alpha(n+1)/\varepsilon$.

Picture:



Proof: Assume r irrational, we can find $(x', y') \in \mathbb{Z}^2$ large enough so that $0 < \alpha y' - rx' < \varepsilon/(n+2)$.

By the assumptions $(2x', 2y'), \dots, ((n+1)x', (n+1)y')$ are also solutions.

In this case, we have that the polynomial

$y'x - x'y$ have $(n+1)$ common zeros.

Hence $(y'x - x'y)$ divides P. (since $\deg P \leq n$)

If we choose ε smaller and repeat the argument $n+1$ times, we would obtain that $\deg P \geq n+1$. Hence, r is rational

Now, assume $r = u/v$ in lowest terms.

Let (x', y') be a solution of $ay - rx = \frac{aj}{v}$.

Then $a(y' + ku) - r(x' + akv) = \frac{aj}{v}$ for any k.

Hence, we conclude that the polynomial

$(ay - rx) - (aj/v)$ divides $P(x,y)$

provided than $aj/v < \varepsilon$ because in such case they share at least $n+1$ zeros and $\deg P \leq n$.

Therefore, we can only have at most n values j for which $aj/v < \varepsilon$. This implies that

$$a(n+1)/v \geq \varepsilon.$$

Hence, $v \leq \dfrac{a(n+1)}{\varepsilon}$ as claimed

Theorem (Rationality Theorem): Let (X,Δ) be a proper klt pair so that $K_X+\Delta$ is not nef. $a \in \mathbb{Z}$ so that $a(K_X+\Delta)$ Cartier. H big & nef Cartier divisor. Define:

nef threshold.

$$r = r(H) := \max\{t \in \mathbb{R} \mid H + t(K_X+\Delta) \text{ is nef}\}.$$

Then r is a rational number of the form u/v $(u,v \in \mathbb{Z})$ where

$$0 < v \leq a(\dim(X)+1).$$

Proof:

Cartier index of $K_X+\Delta$

Step 1: We reduce to the case in which H is bpf.

$$H' = m(cH + da(K_X+\Delta))$$

By bpf Theorem, we know that $|H'|$ is bpf for $m \gg c \gg d > 0$.

$$r(H) = \frac{r(H') + mda}{mc}$$

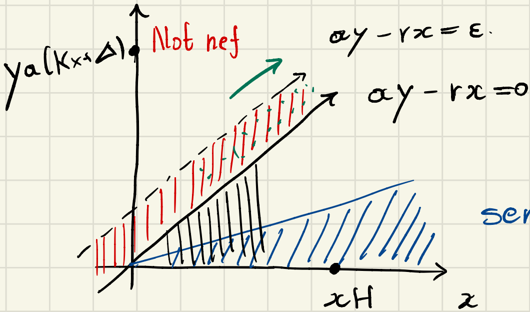
Remark: H is bpf & $H + \varepsilon(K_X+\Delta)$ is semiample

$r(H)$ rational $\iff$ $r(H')$.

If $\operatorname{den}(r(H')) \mid v \implies \operatorname{den}(r(H)) \mid mcv$.

Replace H with H' and now H is bpf.

Step 2: We study the base locus $L(p,q)$.



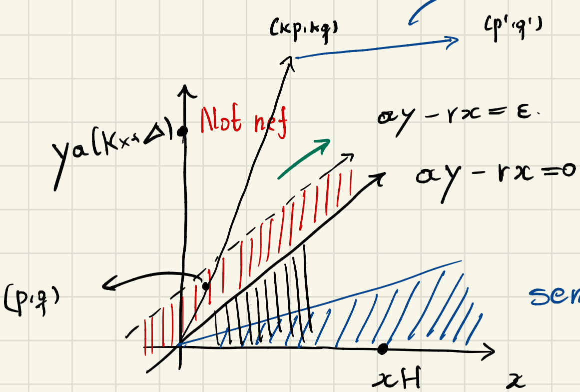
$L(p,q)$ to be the base locus of $|pH + qa(K_X+\Delta)|$.

$L(p,q) = X$ iff $|pH + qa(K_X+\Delta)| = \emptyset$.

(p,q) large enough in the strip, $L(p,q)$ stabilizes

$$L(p',q') \subseteq L(p,q).$$

the semiampleness of this direction gives



By Noetherian induction it stabilizes to L_0.

$I \subseteq \mathbb{Z} \times \mathbb{Z}$ of (p,q), $0 < aq - rp < \varepsilon$ with $L(p,q) = L_0$.

Step 3: We define the polynomial $P(x,y)$ and prove that it does not vanish

$g: Y \longrightarrow X$ a log resolution of (X, Δ).

$D_1 = g^* H$, $D_2 = g^*(a(K_X + \Delta))$, $K_Y = g^*(K_X + \Delta) + A$,

$\lceil A \rceil \geq 0$ g-exc.

$P(x,y) := \chi(xD_1 + yD_2 + \lceil A \rceil)$ is a polynomial of degree $\leq \dim(Y) = \dim(X) = n$.

$y = 0$, $x \gg 0$, D_1 is big & nef.

$P \neq 0$. Furthermore,

$$H^0(Y, pD_1 + qD_2 + \lceil A \rceil) = H^0(X, pH + qa(K_X + \Delta)).$$

(*) From now on, we assume that r is not rational.

Step 4: We show that $L_0 \neq X$.

If $0 < ay - rx < 1$, then.

$$\underbrace{xD_1 + yD_2 + A - K_X}_{\text{big \& nef}} \equiv g^*(xH + \underbrace{(ay-1)(K_X+\Delta)}_{\text{big \& nef}})$$

$H^i(Y, xD_1 + yD_2 + \lceil A \rceil) = 0$ for $i > 0$.

For (p,q) large enough $P(p,q) \neq 0$ by the Lemma 1,

so $|pH + qa(K_X + \Delta)| \neq \emptyset$ for all $(p,q) \in I$,

which means that $L_0 \neq X$, $L_0 \subsetneq X$.

Step 5: We show that $L(p',q') \subsetneq L_0$ for (p',q') large in the strip, leading to a contradiction.

Fix $(p,q) \in I$, $f: Y \longrightarrow (X,\Delta)$ log resolution satisfying:

1) $f^*(\underbrace{pH + (qa-1)(K_X+\Delta)}_{\text{big \& nef}}) - \sum_i' p_j F_j$ ample.

Here, we are using S4

2) $K_Y \equiv f^*(K_X+\Delta) + \sum_i' a_j F_j$ $\quad a_j > -1$ $\longrightarrow$ movable part

3) $f^* |pH + qa(K_X+\Delta)| = |L| + \sum_i' r_j F_j$ $\longrightarrow$ fixed part.

We can choose $c \geq 0$ and $p_j > 0$ so that

$$\sum_i' (-cr_j + a_j - p_j) F_j = A' - F$$

$F \longrightarrow$ prime

$\lceil A' \rceil \geq 0$, A' does not contain F in its support.

F maps to a component B of $L(p,q) = f(\cup_{r_j>0} F_j)$

The base locus of $|pH + qa(K_X+\Delta)|$.

$$N(p',q') = f^*(p'H + q'a(K_X+\Delta)) + A' - F - K_Y$$

$$\equiv \underbrace{\overbrace{c\mathcal{L}}^{\text{nef}} + f^*(pH + (qa-1)(K_X+\Delta)) - \Sigma' p_j F_j}_{\text{ample}}.$$

$$+ f^* \underbrace{((p'-(1+c)p)H + (q'-(1+c)q)a(K_X+\Delta))}_{\text{nef}}$$

We can choose (p',q') with $aq' - rp' < aq - rp$, then

$$(q'-(1+c)q)a < r(p'-(1+c)p), \quad \text{so}.$$

$$(p'-(1+c)p)H + \underbrace{(q'-(1+c)q)a}_{\text{is smaller than nef threshold.}}(K_X+\Delta) \quad \text{is nef.}$$

We conclude that $N(p',q')$ is ample.

$$H^0(Y, f^*(p'H+q'a(K_X+\Delta)) + \lceil A \rceil) \twoheadrightarrow$$

$$H^0(F, (f^*(p'H+q'a(K_X+\Delta)) + \lceil A \rceil)|_F).$$

By adjunction $(f^*(p'H+q'a(K_X+\Delta)) + \lceil A \rceil)|_F =$

$$f^*(p'H + q'a(K_X+\Delta) + A')|_F - K_F.$$

○

Reminder: $K_X + F|_F = K_F$ $\qquad$ $K_X + F + A|_F = K_F + A|_F$

Applying Lemma 1 & Lemma 2 to F, we conclude that

$$H^0(F, (f^*(p'H + q'a(K_X+\Delta)) + \lceil A \rceil)|_F) \neq 0.$$

Hence, $H^0(Y, f^*(p'H + q'a(K_X+\Delta)))$ contains a section $\Gamma \geq 0$ not vanishing at F.

Same argument using neg Lemma implies that Γ actually is disjoint from F. Hence

$0 \leq f_* \Gamma \sim |p'H + q'a(K_X+\Delta)|$ is a section disjoint from $B = f(F) \subseteq L_0$.

Thus, $L(p', q') \subsetneq L_0$ $\longrightarrow\!\longleftarrow$.

So r is rational.

Step 6: We know r is rational, we want to control its denominator.

Assume den is larger than the constant given by Lemma 2

Lemma 2 & $\varepsilon=1$. (p,q) large with $0 < aq - rp < 1$

we have $P(p,q) = h^0(Y, pD_1 + qD_2 + \lceil A \rceil) \neq 0$.

Hence, $|pH + q\alpha(K_X+\Delta)| \neq \emptyset$ for all $(p,q) \in I$.

Choose (p,q) so that $aq - rp$ is the maximum, equal to $\frac{d}{v}$.,

using the notation of step 5, we can show

$$\underline{\chi = h^0 \neq 0} \quad \text{for} \quad (f^*(p'H + q'\alpha(K_X+\Delta)) + \lceil A' \rceil)|_F$$

By Lemma 2, there exists (p',q') large enough in

$0 < \alpha q' - rp' < 1$ with $\varepsilon=1$ and $aq' - rp' < \frac{d}{v} = aq - rp$.

This happens because the later has smaller base locus.

Then, the same argument than step 5 gives us the contradiction $\square$.

Theorem (Non-vanishing): Let X be a proper variety. (X,Δ) a sub-klt pair. D nef Cartier. $aD-(K_X+\Delta)$ nef & big for some $a>0$. Then, for all $m\gg 0$

$$H^0(X, mD-\lfloor\Delta\rfloor)\neq 0.$$

Remark: (X,Δ) klt, $H^0(mD)\neq 0$.

Proof: **Step 1**: Reduce to X smooth & $aD-(K_X+\Delta)$ ample

$f: X' \longrightarrow X$ projective resolution,

$f^*(K_X+\Delta) = K_{X'}+\Delta'$ $\quad (X',\Delta')$ sub-klt pair.

$af^*D-(K_{X'}+\Delta') = f^*(aD-(K_X+\Delta))$ nef & big.

$af^*D-(K_{X'}+\Delta)-\underbrace{F}_{\text{exc. and } \geq 0}$ ample $\quad (X',\Delta'+F)$ sub-klt.

$\Delta''=\Delta'+F$, $\quad f_*(\Delta'')\leq\Delta$ &

$\cancel{0}$ $h^0(X', mf^*D-\lfloor\Delta''\rfloor)\leq h^0(X, mD-\lfloor\Delta\rfloor)$.

Change (X,Δ) with (X',Δ'') $\quad$ X' smooth

D with f^*D $\quad$ $af^*D-(K_{X'}+\Delta'')$ ample.

Step 2: D nef, $D \equiv 0$.

$\lfloor \Delta \rfloor \leq 0$ assume $D \equiv 0$.

KV

$$h^0(X, mD - \lfloor \Delta \rfloor) = \chi(X, mD - \lfloor \Delta \rfloor)$$

$$= \chi(X, -\lfloor \Delta \rfloor)$$

$\chi(D) = \chi(D')$

if $D - D' \equiv 0$

$$= h^0(X, -\lfloor \Delta \rfloor)$$

$\vee$

1.

KV.

D is not numerically trivial. There exists $C \subseteq X$. $D.C > 0$.

Step 3: We claim that there exists q_0 satisfying $x \in X$ not in $\operatorname{supp}(\Delta)$, for $q \geq q_0$ we can find $M(q) \equiv (qD - (K_X + \Delta))$ with $\operatorname{mult}_x M(q) > 2 \dim X$.

for A ample and $e \geq 0$ we have

$$D^e A^{d-e} \geq 0$$

We conclude that:

$$(qD-(K_X+\Delta))^d = (((q-a)D + aD - (K_X+\Delta))^d)$$

$$\geq d(q-a)\,(\underbrace{D\cdot}_{\geq 0}\underbrace{(aD-(K_X+\Delta))^{d-1}}_{\substack{\text{ample.}\\ \text{1-cycle}}})$$

$$(aD-(K_X+\Delta))^{d-1} = C + \text{eff}.$$

where C is the curve satisfying $C.D>0$.

Conclusion: $(qD-(K_X+\Delta))^d \longrightarrow \infty$ if $q \longrightarrow \infty$.

Fact: A ample, for every $Z \subseteq X$ we cand find $\Gamma \sim_{\mathbb{Q}} A$ such that $\operatorname{supp}(\Gamma) \supseteq Z$.

- int with A & Γ is the same

$$|\mathcal{I}_Z(mA)| \ni \Gamma_0, \qquad \Gamma := \Gamma_0/m.$$

X has dim n, $\quad A^{n-1} = \underbrace{\Gamma_1\cdot\Gamma_2\cdots\Gamma_{n-1}}_{\text{contains } C.}$

$$h^0(e(qD-(K_X+\Delta))) \geq \frac{e^d}{d!}(qD-(K_X+\Delta))^d + (\text{lower pow of } e)$$

$M(q,e) \in |e(qD-(K_X+\Delta))|$. imposing that $M(q,e)$ has mult $> 2de$ at x imposes at most

$$\frac{e^d}{d!}(2d)^d + (\text{lower powers of } e)$$

conditions. $q \longrightarrow \infty$, $(qD-(K_X+\Delta))^d > (2d)^d$.

So for q large enough some section satisfies the condition

$$M(q) := M(q,e)/e$$

$M(q) \in |qD-(K_X+\Delta)|$ has mult $> 2d$ at x

Step 4: Consider a log resolution of $(X, \Delta + M(q))$ that dominates $Bl_x X$.

(1) $K_Y \equiv f^*(K_X+\Delta) + \sum' b_j F_j$. $b_j > -1$

(2) $f^*(aD-(K_X+\Delta)) - \sum' p_j F_j$ ample $0 < p_j \ll 1$

(3) $f^* M(q) = \sum' r_j F_j$, F_0 maps to x.

Step 5: We perturb coefficients & lift from lower dim:

$$N(b,c) = bf^*D + \sum_i (-cr_j + b_j - p_j) F_j - K_Y$$

is ample as long as $\frac{1}{2} \geq c$ and $b \geq a + c(g-a)$.

we can always achieve.

Since $x \notin \operatorname{Supp}(\Delta)$, $b_0 = d-1$, $r_0 > 2d$. hence.

$$c < (1 + (d-1) - p_0)/2d < 1/2.$$

$$N(b,c) = bf^*D + A - F - K_Y.$$

we want this $\neq 0$.

$$H^0(Y, bf^*D + \lceil A \rceil - F) = H^0(Y, bf^*D - f^*\lfloor \Delta \rfloor)$$
$$= H^0(X, bD - \lfloor \Delta \rfloor).$$

Since $N(b,c)$ is ample

$$H^1(Y, bf^*D + \lceil A \rceil - F) = H^1(Y, bf^*D + \lceil A - F \rceil) = 0.$$

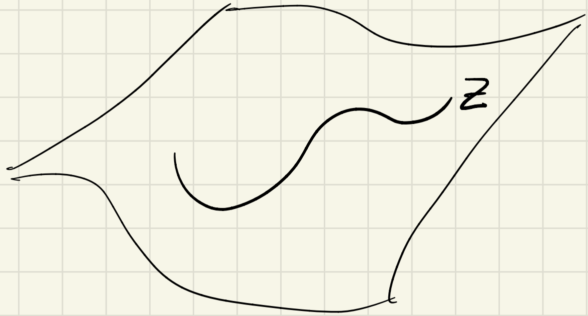
$H^0(X, bD - \lfloor \Delta \rfloor) \neq 0$ provided that

$$H^0(F, (bf^*D + \lceil A \rceil)|_F) \neq 0.$$

this is $\neq 0$

By adjunction & Non-vanishing Thm in dim $d-1$

Idea of all these proofs:



$aD - (K_X + \Delta)$

$\Gamma \in |aD - (K_X+\Delta)|$

bad sing along a subvariety

To adjunction to Z and lift sections from there.

Definition: (X, Δ) log canonical pair.

$Z \subseteq X$ is a log canonical center

$a_E(X, \Delta) = 0$ for some E & $c_E(X) = Z$.

Theorem: $K_X + \Delta|_Z = K_Z + \Delta_Z$

for some $\Delta_Z \geq 0$ s.t. (Z, Δ_Z) lc.

(up to normalizing)